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HYDROMAGNETIC INSTABILITY OF THE MAGNETOSPHERIC BOUNDARY *

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Abstract

We set up a simplified magnetohydrodynamic model of the interface between the solar wind and the geomagnetic field on the tail side of the Earth. Using linearized MHD theory the stability of this situation is discussed. It is shown that unstable waves can exist which travel in the direction of the unperturbed field and grow exponentially with time. The possible acceleration of particles trapped on the magnetic field lines by such unstable waves is discussed in a qualitative manner.

It is also shown that the shortest wavelengths are the most unstable and that as a consequence the analysis should be repeated taking into account both the finite thickness of the interface and also the finite radius of gyration of the particles involved.

I. Introduction

In recent years a great deal of attention has been paid to the geomagnetic tail region in order to explain geomagnetic phenomena. In particular we may mention a paper by Axford and Hines (1961) which concerns a unifying model of high latitude geophysical phenomena.

There are two main reasons for this interest. The first is the need for a mechanism which will produce aurorae continuously. The second is the need for a mechanism which will explain geomagnetic bays and various irregular electron distributions in the ionosphere.

One particular mechanism for producing particle acceleration in the geomagnetic tail has recently been proposed by Gold (1965). He assumes some form of coupling exists between the geomagnetic field in the tail region and the solar wind. As the solar wind flows past the field lines they experience a force which stretches them in the direction of solar wind flow.

As the Earth rotates the field lines rotate and are no longer in contact with the solar wind–geomagnetic field boundary layer. Thus they can 'snap back' into their original configuration with a corresponding energy gain by particles trapped on the field lines. These particles can then be made to produce aurorae.

The main difficulty in Gold's model is that no account is given of the origin and size of the coupling between the solar wind plasma and the geomagnetic field. It has been conjectured that the interaction could be some sort of plasma or MHD wave which corrugates the field lines and produces the required effect.

In this paper we propose to examine the stability of the interface between the solar wind and the geomagnetic field to see if such unstable waves can exist. We will restrict the discussion to the case where magnetohydrodynamics is applicable. In this approximation the boundary layer is taken to be infinitesimal in thickness. We are also interested only in the case where the wavelength of a disturbance is much greater than an ion or electron gyro-radius. In a later paper the equilibrium structure of this boundary layer will be discussed in some detail since it is possible that the existence of a finite thickness layer can produce a drag on the solar wind and a stretching of the field lines.

In our simple model we straighten out the field lines as shown in Figure 1. We assume that on one side of the interface only plasma streaming with the solar wind velocity occurs (region II). We also take the direction of flow of the solar wind to be parallel to the interface. On the other side of the interface (region I) the plasma is assumed to have zero streaming velocity. The plasma in both regions is assumed to be compressible.

2. Equations of Motion

In region I we let the equilibrium values of the streaming velocity, magnetic field and density be $\underset{\sim}{U}_1$, $\underset{\sim}{H}_1$ and ρ_1 respectively. In region II these quantities take on the values $\underset{\sim}{U}_2$, $\underset{\sim}{H}_2$, ρ_2. The Cartesian coordinate system is chosen with the Z-axis normal to the interface and directed from region I to region II.

The actual interface, which we choose to be the plane is defined by the unperturbed pressure condition

$$p_2 + H_2^2/(8\pi) = p_1 + H_1^2/(8\pi) \quad , \tag{1}$$

where p represents gas kinetic pressure.

We now perturb this equilibrium situation by means of an infinitesimal disturbance. Then the velocity, density, pressure and magnetic field on either side of the interface can be written

$$\underset{\sim}{u} = \underset{\sim}{U}_0 + (u, v, w)$$
$$\rho = \rho_0 + \delta\rho$$
$$p = p_0 + \delta p$$
$$\underset{\sim}{H} = \underset{\sim}{H}_0 + (h_x, h_y, h_z)$$

where the subscript '0' represents equilibrium values on either side of this boundary. For the present we will let $\underset{\sim}{U}_0$ and $\underset{\sim}{H}_0$ be 2 -vectors in the (x,y) plane and restrict them later to be vectors in the x-direction as depicted in Figure 1.

We now look for a solution to the linearized equations of motion where all first order perturbation quantities which depend on x, y, z and time, t, vary as

$$f(z) \exp(ik_x x + ik_y y + \omega t) \quad . \tag{2}$$

Should it transpire than an ω exists with a positive real part then in linear theory the interface will be unstable.

Neglecting viscosity it can easily be seen that the linearized equations of motion of the plasma can be written

$$\rho_0(\omega + i\underset{\sim}{k}.\underset{\sim}{U}_0)u = -ik_x\delta p - i\frac{H_{0y}}{4\pi}(k_xh_y - k_yh_x) , \tag{3}$$

$$\rho_0(\omega + i\underset{\sim}{k}.\underset{\sim}{U}_0)v = -ik_y\delta p + i\frac{H_{0x}}{4\pi}(k_xh_y - k_yh_x) , \tag{4}$$

$$\rho_0(\omega + i\underset{\sim}{k}.\underset{\sim}{U}_0)w = -\delta p' - \frac{H_{0x}}{4\pi}(h_x' - ik_xh_z) + \frac{H_{0y}}{4\pi}(ik_yh_z - h_y') , \tag{5}$$

where the prime denotes differentiation with respect to z , i.e.

$$\delta p' \equiv \frac{d}{dz}(\delta p) \quad , \text{and} \quad \underset{\sim}{k} = (k_x, k_y, 0) .$$

The linearized conservation of mass equation demands that

$$(\omega + i\underset{\sim}{k}.\underset{\sim}{U}_0)\delta\rho = -\rho_0(ik_xu + ik_yv + w') . \tag{6}$$

The linearized induction equation can be put in the form

$$(\omega + i\underset{\sim}{k}.\underset{\sim}{U}_0)h_x = i\underset{\sim}{k}.\underset{\sim}{H}_0\,u - H_{0x}(ik_xu + ik_yv + w') \tag{7}$$

$$(\omega + i\underset{\sim}{k}\cdot\underset{\sim}{U}_0)h_y = i\underset{\sim}{k}\cdot\underset{\sim}{H}_0 v - H_{0y}(ik_x u + ik_y v + w') \quad , \tag{8}$$

$$(\omega + i\underset{\sim}{k}\cdot\underset{\sim}{U}_0)h_z = i\underset{\sim}{k}\cdot\underset{\sim}{H}_0 w \, . \tag{9}$$

The solenoidal nature of the magnetic field demands that

$$ik_x h_x + ik_y h_y + h_z' = 0 \quad . \tag{10}$$

Thus it appears as though the set of equations (3) through (10) is complete. However this is not so since (10) can be derived from the induction equation. Thus we need one more equation to complete the set. In particular we require an equation of state which will relate the pressure changes to the density changes.

This can be included in the above set if we let the plasma in regions I and II be characterized by sound speeds C_1 and C_2 respectively. Then an infinitesimal disturbance yields the equation of state

$$\delta p = C_0^2 \, \delta\rho \, . \tag{11}$$

The set (3) through (11) is now complete and particular solutions to the equations can be found once the appropriate boundary conditions are specified. These will be discussed later in the paper once the general solution to the set has been obtained.

From equations (3) and (4) it can be seen that

$$4\pi\rho_0(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)(ik_x u+ik_y v)-(k_x h_y-k_y h_x)(k_x H_{0y}-k_y H_{0x})=k^2\delta p\,, \tag{12}$$

where for brevity we have written $k_x^2+k_y^2=k^2$.

Elimination of δp between (5) and (12) leads to

$$\left[(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)^2+(\underset{\sim}{k}.\underset{\sim}{V}_0)^2\right](w''-k^2w)=(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)^2(ik_x u'+ik_y v'+w'')\,, \tag{13}$$

where the Alfven velocity $\underset{\sim}{V}_0$ is given by

$$\underset{\sim}{V}_0 = \underset{\sim}{H}_0\Big/\sqrt{(4\pi\rho_0)}\quad . \tag{14}$$

By making use of (7), (8), (9), (10), (11), and (13) it can eventually be shown that

$$\begin{aligned}(ik_x u+ik_y v+w')&\left[k^2C_0^2+(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)^2+(\underset{\sim}{k}\times\underset{\sim}{V}_0)^2\right]\\ &=(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)^2w'+(\underset{\sim}{k}.\underset{\sim}{V}_0)\left((\underset{\sim}{k}\times\underset{\sim}{V}_0).\hat{z}\right)(ik_x v-ik_y u)\,,\end{aligned} \tag{15}$$

where $\hat{z}$ is a unit vector in the z -direction.

Use of (3), (4), (7), (8), (9) and (10) shows that

$$\begin{aligned}(ik_x v-ik_y u)&\left[(\omega+i\underset{\sim}{k}.\underset{\sim}{U}_0)^2+(\underset{\sim}{k}.\underset{\sim}{V}_0)^2\right]\\ &=-(\underset{\sim}{k}.\underset{\sim}{V}_0)\left((\underset{\sim}{k}\times\underset{\sim}{V}_0).\hat{z}\right)(ik_x u+ik_y v+w')\,.\end{aligned} \tag{16}$$

Substituting $(ik_x v - ik_y u)$ from (16) into (15) and re-arranging gives

$$(ik_x u + ik_y v + w')\left\{\left[(\omega+i\underset{\sim}{k}.\underset{\sim}{U_0})^2+k^2C_0^2\right]\left[(\omega+i\underset{\sim}{k}.\underset{\sim}{U_0})^2+(\underset{\sim}{k}.\underset{\sim}{V_0})^2\right]+(\omega+i\underset{\sim}{k}.\underset{\sim}{U_0})^2(\underset{\sim}{k}\times\underset{\sim}{V_0})^2\right\}$$

$$= (\omega+i\underset{\sim}{k}.\underset{\sim}{U_0})^2\left[(\omega+i\underset{\sim}{k}.\underset{\sim}{U_0})^2+(\underset{\sim}{k}.\underset{\sim}{V_0})^2\right]w' \quad . \qquad (17)$$

Making use of (17) in (13) enables a differential equation for w to be derived in the form

$$w''\left[(\Omega_0^2+k^2C_0^2)(\Omega_0^2+(\underset{\sim}{k}.\underset{\sim}{V_0})^2)+\Omega_0^2(\underset{\sim}{k}\times\underset{\sim}{V_0})^2-\Omega_0^4\right]$$

$$= k^2 w\left[(\Omega_0^2+k^2C_0^2)(\Omega_0^2+(\underset{\sim}{k}.\underset{\sim}{V_0})^2)+\Omega_0^2(\underset{\sim}{k}\times\underset{\sim}{V_0})^2\right] \quad , \qquad (18)$$

where $\Omega_0 = \omega + i\underset{\sim}{k}.\underset{\sim}{U_0}$.

The solution to (18) can be written as

$$w = A_1 e^{n_1 z} \qquad z<0 \text{ (region I)} \quad , \qquad (19a)$$

$$w = A_2 e^{-n_2 z} \qquad z>0 \text{ (region II)} \quad , \qquad (19b)$$

where n_1 and n_2 are to be found from

$$(n_j/k)^2 = \frac{(\Omega_j^2+k^2C_j^2)(\Omega_j^2+(\underset{\sim}{k}\cdot\underset{\sim}{V_j})^2)+\Omega_j^2(\underset{\sim}{k}\times\underset{\sim}{V_j})^2}{(\Omega_j^2+k^2C_j^2)(\Omega_j^2+(\underset{\sim}{k}\cdot\underset{\sim}{V_j})^2)+\Omega_j^2(\underset{\sim}{k}\times\underset{\sim}{V_j})^2-\Omega_j^4} \tag{20}$$

and $j = 1, 2$ refers to equilibrium values in regions I or II respectively.

The physics of the situation demands that the disturbance field die away as $z \rightarrow \pm \infty$ since it is physically unreasonable that perturbations of the boundary $z = 0$ can influence the plasma behavior increasingly as we move away from this plane.

Thus only those values of n_1 and n_2 are permitted which have $Re(n_1) \geqslant 0$ and $Re(n_2) \geqslant 0$. If either n_1^2 or n_2^2 is real but negative then we must ensure that the disturbance in the appropriate region is a purely progressive wave travelling away from the boundary.

Thus if $Re(n_1) = 0$ then $Im(n_1) \gtrless 0$ according as $Im(\omega) \gtrless 0$, and if $Re(n_2) = 0$ then $Im(n_2) \gtrless 0$ according as $Im(\omega) \gtrless 0$.

Knowing the general solution to the equations of motion the dispersion relation can now be determined once the appropriate boundary conditions are specified.

3. The Dispersion Relation

The appropriate physical boundary conditions which must be applied in our case are:

(i) the normal component of velocity is continuous at the interface: this condition leads to

$$\omega_1 - i\underset{\sim}{k}\cdot\underset{\sim}{U}_1\,\xi = \omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{U}_2\,\xi = \omega\,\xi$$

where ξ is the infinitesimal displacement of the interface.

(ii) the normal component of magnetic field is continuous across the boundary. Using (i) it can easily be shown that this condition is automatically satisfied.

(iii) the normal stress is continuous across the boundary,

i.e. $4\pi(\delta p_1 - \delta p_2) + \underset{\sim}{H}_1\cdot\underset{\sim}{h}_1 - \underset{\sim}{H}_2\cdot\underset{\sim}{h}_2 = 0$.

Use of (1) in (19) yields

$$A_1\Omega_1 = A_2\Omega_2 \quad . \tag{21}$$

Use of (iii) together with (21) yields a dispersion relation which, after a little algebra, can be put in the form

$$\rho_2 n_1\left(\Omega_2^2 + (\underset{\sim}{k}\cdot\underset{\sim}{V}_2)^2\right) + \rho_1 n_1\left(\Omega_1^2 + (\underset{\sim}{k}\cdot\underset{\sim}{V}_1)^2\right) = 0. \tag{22}$$

By inspection of (20) and (22) it is immediately apparent that the dispersion relation is exceedingly difficult to handle in full generality. As a consequence we will consider some simplifying circumstances. We assume that the solar wind streaming velocity, $\underset{\sim}{U}_2$, is everywhere parallel to $\underset{\sim}{H}_1$ and both are in the x-direction only. We further assume that the solar wind occupying region II does not possess an embedded magnetic field and that the plasma in region I does not have a streaming velocity, i.e. $\underset{\sim}{H}_2 = 0$ and $\underset{\sim}{U}_1 = 0$.

4. Applications

Even with the above simplifications waves propagating in the (x,y) plane at some general angle to the magnetic field are difficult to discuss. We will therefore restrict our attention to two fairly simple cases: (a) Transverse propagation, when $\underset{\sim}{k}$ is perpendicular to $\underset{\sim}{H}_1$ and also to $\underset{\sim}{U}_2$, (b) Parallel propagation, when $\underset{\sim}{k}$ is parallel to both $\underset{\sim}{H}_1$ and $\underset{\sim}{U}_2$. While these two cases are rather extreme it is felt that some general idea of the conditions required for instability can be obtained by their consideration.

(a) Transverse propagation.

In this case we can write that

$$\underset{\sim}{k}.\underset{\sim}{V}_1 = 0 ; \quad \underset{\sim}{k}.\underset{\sim}{U}_2 = 0 ; \quad (\underset{\sim}{k}\wedge\underset{\sim}{V}_1)^2 = k^2 V_1^2 ; \quad k^2 = k_y^2 .$$

Since we also have that $\underset{\sim}{U}_1 = 0 = \underset{\sim}{H}_2$ it is a trivial matter to show that

$$n_1^2 = k^2 + \omega^2/(c_1^2 + V_1^2) , \tag{23}$$

$$n_2^2 = k^2 + \omega^2/c_2^2 . \tag{24}$$

The dispersion relation can also be considerably simplified in this case and may be written

$$\rho_2 n_1 = -\rho_1 n_2 . \tag{25}$$

Upon squaring both sides of (25) and making use of (23) and (24) we see that (25) becomes

$$\omega^2 = \frac{k^2 C_2^2 (C_1^2 + V_1^2)(\rho_2^2 - \rho_1^2)}{[\rho_1^2 (C_1^2 + V_1^2) - \rho_2^2 C_2^2]} . \tag{26}$$

By use of (1) and also by letting $C_0^2 = \gamma p_0/\rho_0$ and assuming γ is the same in both regions I and II, it can easily be seen from (26) that the necessary condition for unstable transverse propagating waves can be written

$$0 < \rho_2/\rho_1 < 1 + \frac{(1 - \frac{1}{2}\gamma)}{2\gamma} \frac{H_1^2}{8\pi p_2} . \tag{27}$$

To obtain some idea of the numerical range of values allowed to ρ_2/ρ_1 we consider the case where $8\pi p_1 << H_1^2$ so that the pressure balance at the interface is supplied solely by the solar wind and the geomagnetic field. Since the temperature of the terrestrial plasma does not much exceed 10^4 °K and since satellite measurements in the tail indicate that $\rho_1 \simeq 10^{-23}$ gm. cm^{-3} we can use the above approximation provided $H_1 \gtrsim 2.10^{-5}\ \Gamma$.

We have already neglected the interplanetary magnetic field which is of this order so that we are justified in neglecting $8\pi p_1$ compared with H_1^2 since our calculation is only valid for such a case.

Then

$$8\pi p_2 \simeq H_1^2 \quad ,$$

and we see that (27) becomes

$$0 < \rho_2/\rho_1 < 1 + (2-\gamma)/(4\gamma) \quad . \tag{28}$$

Setting $\gamma = 5/3$ this condition becomes

$$0 < \rho_2/\rho_1 < 1.05 \tag{29}$$

In effect this condition says that the transverse waves can be unstable only if solar wind density is less than the terrestrial plasma density. While the above condition is necessary for instability it is by no means sufficient since in deriving (29) we squared (25) which introduces spurious roots. We can see from (23) and (24) that if $\omega^2 > 0$ then both n_1 and n_2 are positive definite. However in such a case we cannot satisfy (25) which is the correct dispersion relation and hence we conclude that no transverse unstable propagating waves can exist.

Let us now consider the case of propagation where the wave vector is parallel to the magnetic field direction.

(b) Parallel propagation

In this case we can write

$$\underset{\sim}{k} \cdot \underset{\sim}{V_1} = kV_1 \; ; \; \underset{\sim}{k} \cdot \underset{\sim}{U_2} = kU_2 \; ; \; \underset{\sim}{k} \wedge \underset{\sim}{V_1} = 0 \; ; \; k^2 = k_x^2 .$$

Since we still impose the conditions that $\underset{\sim}{U_1} = 0 = \underset{\sim}{H_2}$ we have

$$n_1^2 = \frac{(\omega^2 + k^2V_1^2)(\omega^2 + k^2C_1^2)}{\omega^2(V_1^2 + C_1^2) + k^2C_1^2V_1^2} \tag{30}$$

and

$$n_2^2 = k^2 + (\omega + ikU_2)^2 C_2^{-2} . \tag{31}$$

The dispersion relation can be written

$$\rho_2 n_1 (\omega + ikU_2)^2 = -n_2 \rho_1 (\omega^2 + k^2V_1^2) . \tag{32}$$

Squaring both sides of (32) and making use of (30) and (31) enables us to write that

$$\begin{aligned} &y^6(C_2^2\sigma^{-2} - C_1^2 - V_1^2) + 2iUy^5(2C_2^2\sigma^{-2} - V_1^2 - C_1^2) + y^4\begin{bmatrix} C_2^2\sigma^{-2}(C_1^2 - 6U^2) - V_1^2(V_1^2 + 2C_1^2) \\ -(V_1^2 + C_1^2)(C_2^2 - U^2) \end{bmatrix} \\ &+ 2iUy^3\left[2C_2^2\sigma^{-2}(C_1^2 - U^2) - V_1^2(V_1^2 + 2C_1^2)\right] + y^2\begin{bmatrix} C_2^2\sigma^{-2}U^2(U^2 - 6C_1^2) - C_1^2V_1^4 \\ -V_1^2(V_1^2 + C_1^2)(C_2^2 - U^2) \end{bmatrix} \\ &- 2iUC_1^2y(V_1^4 + 2U^2C_2^2\sigma^{-2}) + C_1^2\left(C_2^2U^4\sigma^{-2} - V_1^4(C_2^2 - U^2)\right) = 0 . \end{aligned} \tag{33}$$

where we have dropped the subscript on the solar wind flow velocity. For convenience we have also set

$$y = \omega/k \quad \text{and} \quad \sigma = \rho_2/\rho_1 .$$

It is well known that it is impossible to solve analytically polynomials of higher degree than a quartic. Thus in order to proceed beyond this point we must make a further approximation. The simplest situation to consider is the case in which the terrestrial plasma is taken to be completely cold. While this is clearly a gross oversimplification it does at least have the advantage of making the polynomial tractable without reducing the physics of the situation too seriously.

Making this assumption we see that $C_1 = 0$ and we then have

$$\rho_2 = H_1^2/(8\pi) . \tag{34}$$

As a result equation (33) becomes

$$y^4\sigma(\sigma - 2/\gamma) + 4iU\sigma y^3(\sigma - 1/\gamma) - 2\sigma y^2\left[U^2(3\sigma - 1/\gamma) + C_2^2\gamma^{-1}(2\sigma/\gamma + 1)\right]$$
$$- 4iU\sigma^2 y(U^2 + 2C_2^2\gamma^{-2}) + \sigma^2\left(U^4 - 4C_2^2\gamma^{-2}(C_2^2 - U^2)\right) = 0, \tag{35}$$

where use has been made of (34) in the form

$$\gamma V_1^2 = 2\sigma C_2^2 . \tag{36}$$

While it is possible to solve (35) analytically without making any further approximation, the final values of y are algebraically so complicated that further assumptions would have to be made at this stage about the relative magnitude of the various parameters in (35). We choose to proceed by making assumptions about the size of various parameters and then find the roots of the quartic.

We will make the assumption that $\gamma\sigma \gg 2$. We expect that σ will indeed be much larger than unity in practice.

We then have to decide whether the ratio of the solar wind velocity to its sound speed is small or large on the tail side of the Earth between the stand-off shock and the magnetopause boundary. If we assume that the shock is strong and remains so even well away from the sub-solar point then it can easily be shown by a consideration of the Rankine-Hugoniot conditions that $U^2 \ll C_2^2$. However it has recently been pointed out to the author (Jokipii, 1965) that the shock may be fairly weak at high latitudes. Should this be the case then we have that the solar wind is virtually unaltered in passing through a very weak shock. Hence we may have $U \gg C_2$. We will discuss each of these cases in turn.

Let us first consider the case $\gamma\sigma \gg 2$ and $U \ll C_2$. In this case (35) becomes

$$x^4 + i\left(2\gamma U C_2^{-1}\right)x^3 - x^2 - i\left(\gamma U C_2^{-1}\right)x - \gamma^2/4 = 0 , \qquad (37)$$

where

$$y = 2C_2\gamma^{-1}x .$$

The roots of (37) are given approximately by

$$x_{1,2} \simeq \pm\sqrt{\left(\frac{1+\sqrt{(1+\gamma^2)}}{2}\right)} \mp \frac{i\gamma U}{2C_2\sqrt{\left(\frac{1+\sqrt{(1+\gamma^2)}}{2}\right)}}, \tag{38a}$$

$$x_{3,4} \simeq \pm i\sqrt{\left(\frac{\sqrt{(1+\gamma^2)}-1}{2}\right)} \mp \frac{\gamma U}{2C_2\sqrt{\left(\frac{\sqrt{(1+\gamma^2)}-1}{2}\right)}}. \tag{38b}$$

Of the four above roots it can be seen that only x_1 or x_4 can give rise to unstable waves. Consider first x_1, then we have that

$$\omega \simeq 2kC_2\gamma^{-1}\sqrt{\left(\frac{1+\sqrt{(1+\gamma^2)}}{2}\right)} - \frac{ikU}{\sqrt{\left(\frac{1+\sqrt{(1+\gamma^2)}}{2}\right)}} \tag{39}$$

With this value of ω it can easily be verified that

$$n_1 \simeq k\left[1+(\sigma\gamma)^{-1}\left(1+\sqrt{(1+\gamma^2)}\right)\right]^{1/2}\left[1 - \frac{i\gamma U 2\left(\frac{1+\sqrt{(1+\gamma^2)}}{2}\right)^{-1/2}}{C_2\left(\sqrt{(1+\gamma^2)}-1\right)\left(\sigma\gamma+1+\sqrt{(1+\gamma^2)}\right)\sqrt{(1+\gamma^2)}}\right], \tag{40}$$

$$n_2 \simeq k\left[1+2\gamma^{-2}\left(1+\sqrt{(1+\gamma^2)}\right)\right]^{1/2}\left[1 + \frac{iU\left(1+\sqrt{(1+\gamma^2)}\right)}{\gamma C_2\left(\sqrt{(1+\gamma^2)}-1\right)\left(1+2\gamma^{-2}\left(1+\sqrt{(1+\gamma^2)}\right)\right)\sqrt{(1+\gamma^2)}}\right]. \tag{41}$$

From (40) and (41) respectively we see that $Re(n_1) > 0$ and $Re(n_2) > 0$ as required by energy considerations.

From (39) it can be seen that the growth of the wave does not depend, in this approximation, on the solar wind velocity and is determined purely by the sound speed of the solar wind and the geomagnetic field. Likewise the phase velocity of the wave depends only on the fact that the solar wind is streaming.

While the above values satisfy (33) they do not satisfy (32) and hence we conclude that this wave does not in fact exist. Let us therefore consider the other possible unstable wave, namely that arising from x_4. With x_4 as given by (38b) we have that

$$\omega \simeq -i\,2C_2 k\gamma^{-1}\sqrt{\left(\frac{\sqrt{(1+\gamma^2)}-1}{2}\right)} + \frac{Uk}{\sqrt{\left(\frac{\sqrt{(1+\gamma^2)}-1}{2}\right)}} \quad . \tag{42}$$

With this value of ω it can easily be seen that

$$n_1 \simeq k\left[1-(\sigma\gamma)^{-1}\left(\sqrt{(1+\gamma^2)}-1\right)\right]^{1/2}\left[1-\frac{iU}{2\sigma C_2\left(1-(\sigma\gamma)^{-1}\left(\sqrt{(1+\gamma^2)}-1\right)\right)}\right], \tag{43}$$

$$n_2 \simeq k\left[1+2\gamma^{-2}-2\gamma^{-2}\sqrt{(1+\gamma^2)}\right]^{1/2}\left[1-\frac{2iU}{\gamma C_2}\right] . \tag{44}$$

With these values of the parameters we see that (32) is satisfied and hence the wave exists and is unstable with a growth rate given by

$$Re(\omega) = \frac{Uk}{\sqrt{\left(\frac{\sqrt{(1+\gamma^2)}-1}{2}\right)}} \quad . \tag{45}$$

In the approximation we have made the wave's growth depend only on the fact that the solar wind is streaming past the boundary of the magnetosphere.

It can be seen that the shortest wavelengths give rise to the most unstable situation. However since this is purely a magnetohydrodynamic calculation we are restricted to wavelengths much greater than a boundary layer thickness or a particle's gyro-radius, whichever is the greater. So in order to discuss the instability properly, we should use the collisionless Boltzmann equation and take into account both the particles' gyro-radii and the thickness and structure of the boundary layer.

Let us now consider the case where the solar wind is still supersonic so that we can evaluate the roots of (35) under the approximations $\gamma\sigma >> 2$ and $U >> C_2$.

Under these conditions we see that (35) can be written

$$x^4 + 4ix^3 - 6x^2(1 + 2\epsilon/3) - 4ix(1 + 2\gamma^{-2}\epsilon) + 1 + 4\epsilon\gamma^{-2} = 0 , \tag{46}$$

where $y = Ux$ and $\epsilon = C_2^2 U^{-2}$.

The roots of (46) are approximately given by

$$x = -i\left(1 + \frac{C_2^2(1+3\gamma^{-2})}{8U^2}\right), \tag{47}$$

to first order in $C_2^2 U^{-2}$ since to zeroth order in $C_2^2 U^{-2}$ (46) is a perfect quartic.

The point which we wish to make here is that, under the approximations made, the phase velocity of the wave is nearly independent of all parameters except for the solar wind streaming velocity. We also note that in this case no unstable wave exists in the limit of $U >> C_2$.

We have shown that in the limit of $U << C_2$ or $U >> C_2$ the magnetospheric and solar wind interface is unstable or stable respectively to parallel propagating MHD waves provided $\gamma\sigma \gg 2$. The region of marginal stability, when $U \simeq O(C_L)$, will not be discussed since the results are algebraically very complicated.

Despite the fact that the calculation is invalid for small wavelengths we can see qualitatively how such unstable waves can cause particle acceleration. We know that the real field times are not complete'y straight but, in at least one model of the magnetospheric tail, they close in a manner similar to that depicted in figure 2.

Let us assume that the parallel propagation instability develops simultaneously, or nearly so, on either side of the Earth. Then second order non-linear effects will stop the growth of the wave and leave 'bumps' in the magnetic field which will propagate in the anti-solar direction. They travel along the field lines until appreciable bending of the field occurs when we assume the bumps 'slide off' the field lines.

A particle trapped on a field line between two such bumps will be accelerated by them provided its pitch angle satisfies the trapping condition while the bumps travel to the 'slip-off' region.

One of two things can happen to the particle: first it remains trapped until the bumps disappear and then it can travel back along the field line and mirror at a considerably lower altitude in the Earth's atmosphere than the mirroring altitude before the bumps occurred; secondly, because of the gain in parallel momentum, the particle may be able to 'squeeze' through a bump and then follow the field line down to its new, lower altitude, mirroring point.

In either case the mirroring altitude may be reduced to a height where ionization can take place. If we have a sufficient number of particles to which this happens then an aurora is produced.

While the above is only a qualitative prediction, once the equilibrium structure of the boundary layer has been discussed it should be possible to place this argument on a quantitative footing.

5. A Comparison

In a recent paper by Fejer (1964) a similar MHD stability problem to the above is discussed. However the conditions he imposes on either side of the interface are different to those employed in the present calculation. We will compare the results obtained by Fejer with those obtained here for a special case: we suppose there exist parallel magnetic fields of equal magnitude on either side of the boundary between fluids of identical acoustic properties, i.e. $C_1 = C_2$; $\underset{\sim}{H}_1 = \underset{\sim}{H}_2$.

In such a case we see that the dispersion relation for parallel propagation $(k_y = 0)$ can be written

$$n_1\left[(\omega + ikU)^2 + k^2V^2\right] + n^2\left[\omega^2 + k^2V^2\right] = 0 \qquad (48)$$

where

$$n_1^2 = \frac{(\omega^2 + k^2V^2)(\omega^2 + k^2C^2)}{\omega^2(V^2 + C^2) + k^2V^2C^2} \qquad (49a)$$

$$n_2^2 = \frac{\left[(\omega+ikU)^2+k^2V^2\right]\left[(\omega+ikU)^2+k^2C^2\right]}{(\omega+ikU)^2(V^2+C^2)+k^2V^2C^2} \; . \tag{49b}$$

Writing $\omega = -ik\varpi$, we see that upon squaring both sides of (48) we have

$$(C^2-\varpi^2)\left[V^2-(U-\varpi)^2\right]\left[V^2C^2-(U-\varpi)^2(V^2+C^2)\right]^2$$
$$= (V^2-\varpi^2)^2\left[C^2-(U-\varpi)^2\right]\left[V^2C^2-\varpi^2(V^2+C^2)\right] \; . \tag{50}$$

An inspection of this equation shows that the root $\varpi = V$ becomes a double root when

$$\varpi - V = \varpi - U + \frac{VC}{\sqrt{(V^2+C^2)}}$$

or

$$C = V(U-V)\left(2VU-U^2\right)^{-1/2} \; , \tag{51}$$

which is identical to Fejer's condition (13). For instability we require $V > f(C,U)$ where $f(C,U)$ is the function determined from (51).

The presence of a double root in the square of the dispersion relation is clearly seen to be the requirement for marginal stability. Thus in this case we see that our analysis and that of Fejer are identical.

A second case considered by Fejer is that where $C_1 = C_2$ but the equilibrium magnetic field is taken to be perpendicular to the boundary and of equal magnitude on both sides. It is difficult to see how such an equilibrium can be maintained since the flow of plasma parallel to the interface on one side will cause a shearing of the magnetic field which should be included in describing the equilibrium situation.

While realizing that Fejer's calculation only aimed to show how an unstable situation could develop we feel that the choice of parameters made in this paper probably give a better approximation to conditions in the magnetospheric tail near the interface than those presented by Fejer.

6. Conclusion

Using an extremely simplified model of the solar wind–magnetosphere boundary in the geomagnetic tail it has been shown that unstable parallel propagation modes can arise when magnetohydrodynamics is assumed to be valid. The most rapidly growing mode is shown to be that with the shortest wavelength.

It is demonstrated qualitatively that such waves can give rise to particle acceleration and as a consequence may be a contributory factor in auroral production.

The main conclusion, which we wish to emphasize, is that both the above analysis and Fejer's calculations can aid only as indications of the correct situation since we are limited to wavelengths much greater than a boundary layer thickness and a particle's gyro-radius. Thus the argument should be repeated using the Vlasov equation and taking into account the equilibrium structure of the boundary layer.

We also wish to point out that short wavelength plasma instabilities are probably capable of producing the coupling between the solar wind and the magnetospheric boundary required in both Axford and Hines' and Gold's models.

Certainly the calculations based on linearized MHD equations indicate that processes occurring on a scale length of a boundary layer thickness or a gyro-radius are candidates for producing such a coupling.

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References

Axford, W. I. and Hines, C. O. 1961, Can. J. Phys. 39, 1433.

Gold, T. 1965, Quat. J.R.A.S. (in press).

Fejer, J. A. 1964, Phys. Fluids 7, 499.

Jokipii, J. R. 1965, private communication.

Figure Captions

Figure 1. The equilibrium model.

Figure 2. Particle acceleration mechanism.

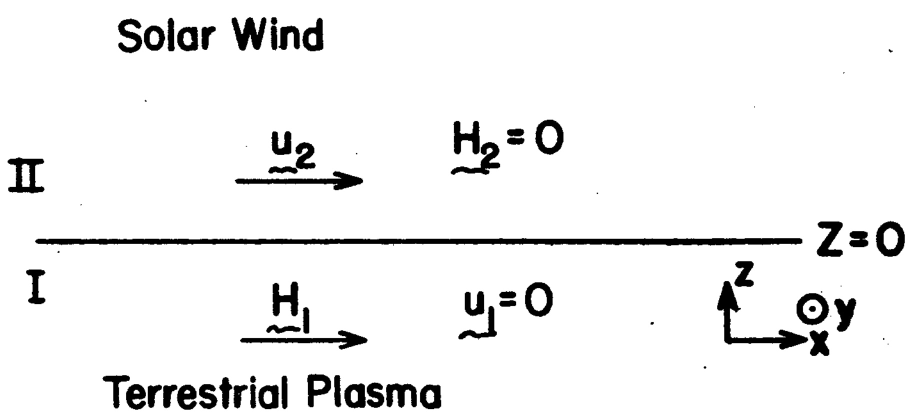
Solar Wind
$\underset{\sim}{u}_2$
$\underset{\sim}{H}_2 = 0$
II
$Z=0$
I
$\underset{\sim}{H}_1$
$\underset{\sim}{u}_1 = 0$
z
y
x
Terrestrial Plasma

Fig. 1

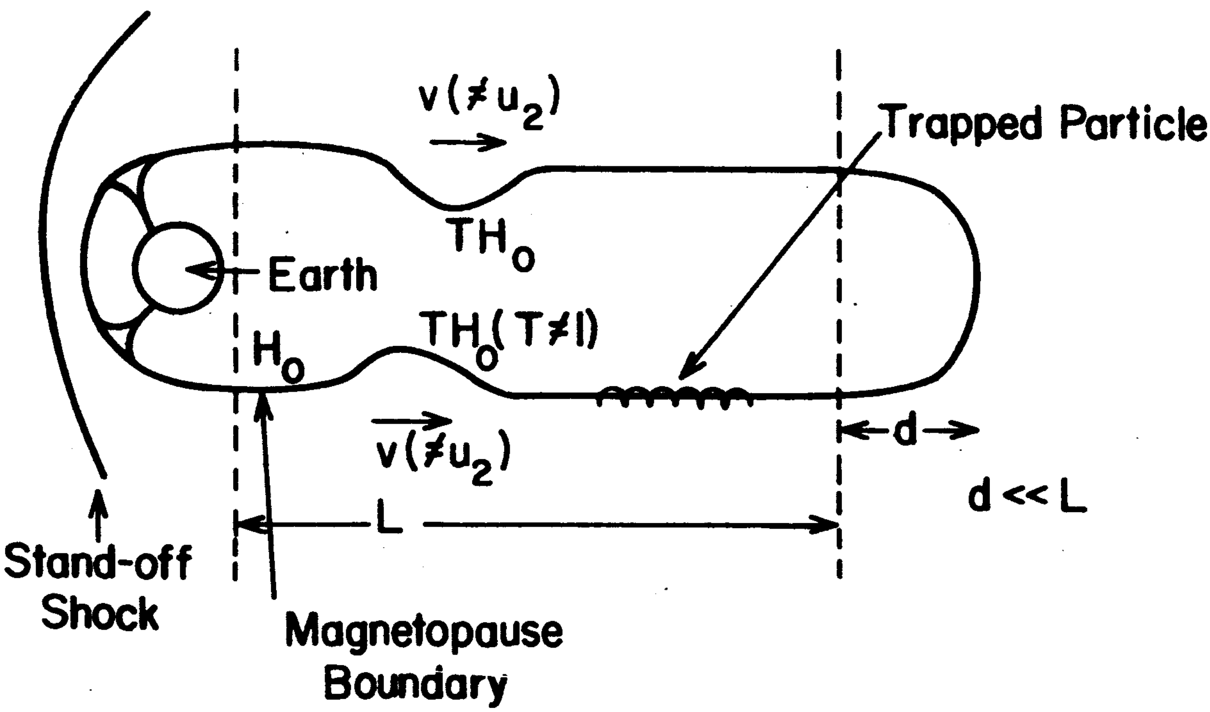
v(≠u2)
Trapped Particle
TH0
Earth
TH0 (T≠1)
H0
v(≠u2)
d
d << L
L
Stand-off
Shock
Magnetopause
Boundary

Fig. 2